ON THE $T(q)$-CONDITIONS OF SMALL CANCELLATION THEORY

BY

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ABSTRACT

In this paper we give a graphical method which can be used to determine whether or not a group presentation satisfies the small cancellation condition $T(q)$. We use this method to determine all 2- and 3-generator presentations satisfying $T(4)$.

1. Introduction

The small cancellation conditions $C(p)$ and $T(q)$ arise naturally when one uses van Kampen diagrams to study groups given by presentations. A number of results have been found for groups defined by $C(p)$, $T(q)$ presentations where $1/p + 1/q = 1/2$ (that is, where $(p,q) = (6,3), (4,4), (3,6)$) [2–5]. It is therefore useful to be able to recognise when a presentation satisfies the conditions $C(p)$, $T(q)$. In this paper we describe a graphical method which can be used to determine whether or not a presentation satisfies $T(q)$.

The values of q for which $T(q)$ is of interest are 3, 4, 6. The condition $T(3)$ is no restriction. The $T(6)$ condition will be discussed in another paper by the second author. Our principal concern in this paper is with the $T(4)$ condition. We determine all 2- and 3-generator presentations satisfying $T(4)$. We also give enough graphical information to enable the reader to determine all 4-generator presentations satisfying $T(4)$ if he/she so wishes.

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2. Star complexes and the $T(q)$ condition

2.1. 1-Complexes

A 1-*complex* consists of two disjoint subsets V (vertices), E (edges) together with functions $\iota : E \to V$, $\tau : E \to V$, $^{-} : E \to E$ satisfying: $\iota(e) = \tau(\bar{e})$, $\bar{\bar{e}} = e$, $\bar{e} \neq e$ for all $e \in E$. A *closed path* in this 1-complex is a succession of edges $e_1, e_2, \ldots, e_m$ with $\tau(e_i) = \iota(e_{i+1})$ for $i = 1, \ldots, m$ (where subscripts are computed mod m). The closed path is said to be *reduced* if $e_{i+1} \neq \bar{e}_i$ for $i = 1, \ldots, m$. We call m the *length* of the path. A closed path of length 1 is called a *loop*. For a vertex v of a 1-complex, the set of edges e with $\iota(e) = v$ is denoted by Star(v).

We remark that 1-complexes are often called graphs by combinatorial group theorists (for example, in [6]). However, we reserve the term graph for a different concept, namely the following: a *graph* consists of a set V (vertices) together with a set of two-element subsets of V (edges). Our basic reference for graph theory will be [1].

Given any 1-complex $\mathscr{X}$ (with vertex set V, say), we can associate with it a graph $\Gamma(\mathscr{X})$ as follows. The vertex set of $\Gamma(\mathscr{X})$ is V; $\{u, v\}$ is an edge of $\Gamma(\mathscr{X})$ if and only if there is an edge e of $\mathscr{X}$ with $\iota(e) = u$, $\tau(e) = v$. For each ordered pair (u, v) of elements of V we define

$$m_{u,v} = |\{e : e \text{ is an edge of } \mathscr{X}, \iota(e) = u, \tau(e) = v\}|.$$

Note that $m_{u,v} = m_{v,u}$. If Γ is any graph with vertex set V then we can define a *weight function* on Γ by assigning weight $m_{u,v}$ to the edge $\{u, v\}$ (assuming, of course, that the $m_{u,v}$ are finite). The *weight of a vertex* of Γ is then defined to be the sum of the weights of the edges incident with the given vertex (provided the sum exists). Note that if Γ contains $\Gamma(\mathscr{X})$ then the weight (if it exists) of a vertex v of Γ is $|\text{Star}(v)| - m_{v,v}$.

2.2. A Formulation of the $T(q)$ Condition

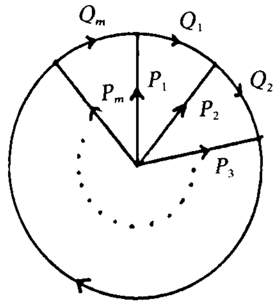
A set $\boldsymbol{r}$ of words on an alphabet $\boldsymbol{x}$ is said to be *symmetrized* provided:

(i) each element of $\boldsymbol{r}$ is non-empty and cyclically reduced;

(ii) if $R \in \boldsymbol{r}$ then all cyclic permutations of $R^{\pm 1}$ belong to $\boldsymbol{r}$.

(We note for future reference that if $\boldsymbol{s}$ is a set satisfying (i) then the set of all cyclic permutations of elements of $\boldsymbol{s} \cup \boldsymbol{s}^{-1}$ is a symmetrized set, called the *symmetrized closure* of $\boldsymbol{s}$.) A group presentation $\langle \boldsymbol{x} ; \boldsymbol{r} \rangle$ is said to be symmetrized if $\boldsymbol{r}$ is symmetrized.

Consider the symmetrized presentation $\mathscr{P} = \langle \boldsymbol{x} ; \boldsymbol{r} \rangle$. A *wheel* (over $\mathscr{P}$) is a reduced van Kampen diagram of the form

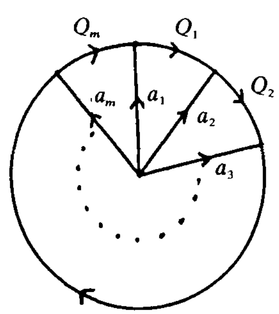


where $m \geqq 3$. Here the P_i and Q_i are non-empty words and $P_iQ_iP_{i+1}^{-1} \in \boldsymbol{r}$ for $i = 1, \ldots, m$ (subscripts are computed mod m). We say that the wheel has m *spokes*. If q is an integer greater than 2, then the $T(q)$ condition asserts that *every wheel over* $\mathcal{P}$ *has at least* q *spokes*. We wish to give an alternative formulation of this condition.

We associate with $\mathcal{P}$ a 1-complex $\mathcal{P}^{st}$ (called the *star complex* of $\mathcal{P}$) as follows. The vertex set is $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$, and the edge set is $\boldsymbol{r}$. For a given edge R of $\mathcal{P}^{st}$ we define $\iota(R)$ to be the first symbol of R, and $\tau(R)$ to be the inverse of the last symbol of R. We define $\bar{R}$ to be R^{-1}. Note that since each element of $\boldsymbol{r}$ is cyclically reduced, $\mathcal{P}^{st}$ has no loops.

Now observe that our wheel above gives rise to a reduced closed path of length m in $\mathcal{P}^{st}$, namely the path $P_1Q_1P_2^{-1}, \ldots, P_mQ_mP_1^{-1}$.

Conversely, consider a reduced closed path $\alpha = R_1, R_2, \ldots, R_m$ ($m \geqq 3$) in $\mathcal{P}^{st}$. If each R_i has length at least 3, then α gives rise to a wheel with m spokes. For let $R_1 = a_1Q_1a_2^{-1}$ where $a_1, a_2 \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}$. Since $\tau(R_1) = \iota(R_2)$ we have $R_2 = a_2Q_2a_3^{-1}$ where $a_3 \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}$. Then $R_3 = a_3Q_3a_4^{-1}$ where $a_4 \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}$, and so on. Since α is closed we eventually obtain $R_m = a_mQ_ma_1^{-1}$. Thus we have the wheel



(Note that this van Kampen diagram is reduced since α is reduced.)

If some term R_i of the path α above has length less than 3, then α does not give rise to a wheel. However, this is a very pathological situation, for each letter of R_i will be a piece, and so, in particular, $\mathcal{P}$ will not even satisfy the small cancellation condition $C(3)$. Since $T(q)$ is only useful when considered in conjunction with $C(p)$ with $p \geqq 3$, this pathological situation will not occur in practice.

We see from the above discussion that, apart from the pathology described in the previous paragraph, $T(q)$ is equivalent to:

(*) *there is no reduced closed path in $\mathcal{P}^{st}$ of length m, where $3 \leqq m < q$.*

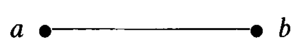
We will take (*) to be the definition of $T(q)$ for any symmetrized presentation $\mathcal{P}$.

2.3. On Star Complexes and their Associated Graphs

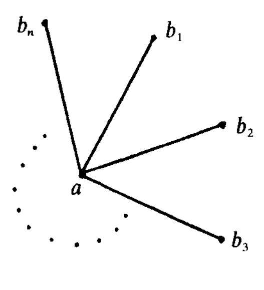
We note some facts about the star complex $\mathcal{P}^{st}$, and the associated graph $\Gamma(\mathcal{P}^{st})$, of a symmetrized presentation $\mathcal{P} = \langle \boldsymbol{x}; \boldsymbol{r} \rangle$.

We remark that star complexes have been considered by other authors (see [4]). They will be discussed in a wider context in a forthcoming article by S. J. Pride.

2.3.1. Consider $\Gamma(\mathcal{P}^{st})$. In passing from $\mathcal{P}^{st}$ to $\Gamma(\mathcal{P}^{st})$ some information is lost. Information not lost is which elements of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ are predecessors of a given element of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ in relators of $\mathcal{P}$. Thus, if we have an edge



in $\Gamma(\mathcal{P}^{st})$, then this tells us that in at least one relator, a is preceded by b^{-1}. Consequently, if we look at all edges of $\Gamma(\mathcal{P}^{st})$ incident with a given vertex a



then in relators of $\mathcal{P}$, a is preceded by precisely $b_1^{-1}, b_2^{-1}, \ldots, b_n^{-1}$.

2.3.2. It is clear that if $\boldsymbol{x}$ is finite then there is a finite symmetrized subpresentation $\mathscr{P}_0 = \langle \boldsymbol{x}; \boldsymbol{r}_0 \rangle$ ($\boldsymbol{r}_0 \subseteq \boldsymbol{r}$) of $\mathscr{P}$ such that $\Gamma(\mathscr{P}_0^{st}) = \Gamma(\mathscr{P}^{st})$. This observation will enable us to restrict attention to finite presentations.

2.3.3. Suppose the presentation $\mathscr{P}$ is finite. Let Γ be a graph containing $\Gamma(\mathscr{P}^{st})$ and having the same vertex set, and consider the weight function on Γ as described in §2.1. Then the weight of a vertex x is equal to the weight of x^{-1}. For since $\mathscr{P}^{st}$ has no loops, the weights of x and x^{-1} are $|\mathrm{Star}(x)|$, $|\mathrm{Star}(x^{-1})|$ respectively. Let $R \in \mathrm{Star}(x)$, so that $R = xS$ for some word S. Then $x^{-1}S^{-1} \in \mathrm{Star}(x^{-1})$. This gives a bijection from $\mathrm{Star}(x)$ to Star (x^{-1}).

3. On $T(4)$ presentations

3.1. Some Definitions

In order to state our results we need to make a number of definitions.

Let $\boldsymbol{x}$ be an alphabet. A word on $\boldsymbol{x}$ is said to be *positive* (resp. *negative*) if only positive (resp. negative) powers of elements of $\boldsymbol{x}$ appear in the word. A word is said to be *square-free* if it has no subword of the the form x^k with $x \in \boldsymbol{x}$ and $|k| > 1$. A word is said to be *cyclically square-free* if it has length greater than 1 and if all its cyclic permutations are square-free. Let $\boldsymbol{w}_{-1}$, $\boldsymbol{w}_1$ be disjoint sets of non-empty words on $\boldsymbol{x}$. A word $t_1t_2\cdots t_s$ ($s \geqq 1$, $t_i \in \boldsymbol{w}_{-1} \cup \boldsymbol{w}_1$ for $i = 1, \ldots, s$) is called an *alternating* $(\boldsymbol{w}_{-1}, \boldsymbol{w}_1)$*-word* if the following holds: for $1 \leqq i < s$, if $t_i \in \boldsymbol{w}_\varepsilon$ ($\varepsilon = \pm 1$) then $t_{i+1} \in \boldsymbol{w}_{-\varepsilon}$. The word is called a *cyclically alternating* $(\boldsymbol{w}_{-1}, \boldsymbol{w}_1)$*-word* if it is an alternating $(\boldsymbol{w}_{-1}, \boldsymbol{w}_1)$-word with the additional property that t_1 and t_s do not belong to the same set $\boldsymbol{w}_{-1}$ or $\boldsymbol{w}_1$. The t_i are called the *factors.* When considering a factor t_i of a cyclically alternating $(\boldsymbol{w}_{-1}, \boldsymbol{w}_1)$-word $t_1t_2\cdots t_s$, we will sometimes be interested in the adjacent factors t_{i-1}, t_{i+1}. In this regard, it is to be understood that subscripts are computed mod s, so that if $i = 1$ then $t_{i-1} = t_s$, while if $i = s$ then $t_{i+1} = t_1$.

The *extended symmetric group* (on $\boldsymbol{x}$) is the group of permutations of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ generated by the permutations $(xy)(x^{-1}y^{-1})$, (xx^{-1}) ($x, y \in \boldsymbol{x}$). We denote this group by Ω_n where $n = |\boldsymbol{x}|$. The group Ω_n induces a group of automorphisms (also denoted Ω_n) of the free group on $\boldsymbol{x}$.

Let Γ be a graph with vertex set V, and let X be a set with $|X| = |V|$. A *labelling of* Γ (by X) is a graph with vertex set X and edge set

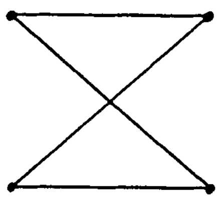
$$\{\{\lambda(u), \lambda(v)\} : \{u, v\} \text{ is an edge of } \Gamma\},$$

where $\lambda : V \to X$ is some bijection. If Σ is a group of permutations of X, then two labellings of Γ, with underlying bijections λ, λ' say, are said to be Σ*-equivalent* if $\lambda = \sigma\lambda'$ for some $\sigma \in \Sigma$.

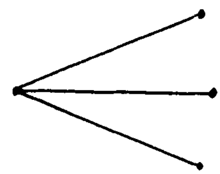
3.2. Two-Generator Presentations

We will show that *a symmetrized presentation* $\mathscr{P} = \langle a, b\,; \boldsymbol{r}\rangle$ *satisfies* $T(4)$ *if and only if either*: *each element of* $\boldsymbol{r}$ *is cyclically square-free*; *or, there is an automorphism* $\sigma \in \Omega_2$ *such that each element of* $\sigma\boldsymbol{r}$ *is positive or negative.*

For $\mathscr{P}$ satisfies $T(4)$ if and only if $\Gamma(\mathscr{P}^{st})$ is a subgraph of a labelling by $\{a, a^{-1}, b, b^{-1}\}$ of one of:

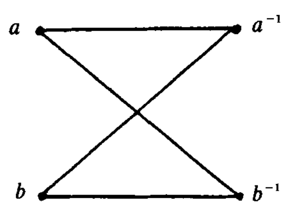


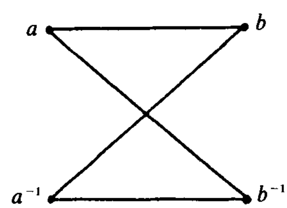
$K_{2,2}$



$K_{1,3}$

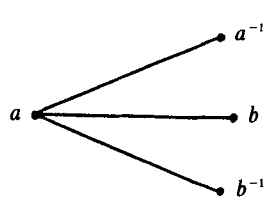
Up to Ω_2-equivalence, the labellings of $K_{2,2}$ are:





Now $\Gamma(\mathscr{P}^{st})$ will be a subgraph of the former labelling if and only if each relator of $\mathscr{P}$ is positive or negative; $\Gamma(\mathscr{P}^{st})$ will be a subgraph of the latter labelling if and only if each relator of $\mathscr{P}$ is cyclically square-free.

Up to Ω_2-equivalence, the only labelling of $K_{1,3}$ is:



Suppose $\Gamma(\mathscr{P}^{st})$ is a subgraph of this graph. If the vertex a^{-1} of $\Gamma(\mathscr{P}^{st})$ is isolated then so is a, in which case all vertices are isolated, and $\mathscr{P}$ has no relators. If the edge $\{a, a^{-1}\}$ belongs to $\Gamma(\mathscr{P}^{st})$ then a^{-1} is preceded by a^{-1} (and only by a^{-1}) in relators of $\mathscr{P}$. Thus neither of the edges $\{a, b\}$, $\{a, b^{-1}\}$ belongs to $\Gamma(\mathscr{P}^{st})$, so $\boldsymbol{r}$ consists solely of powers of a.

3.3. Three-Generator Presentations

We will show that *a symmetrized presentation* $\mathcal{P} = \langle a, b, c\,; \mathbf{r}\rangle$ *satisfies* $T(4)$ *if and only if there is an automorphism* $\sigma \in \Omega_3$ *such that* $\mathbf{r}$ *is the symmetrized closure of a set* σs, *where* s *satisfies one of the seven conditions 3.3.1–3.3.7 listed below.*

3.3.1. Each element of s is positive.

3.3.2. Each element of s is either a power of c, or a cyclically square-free word in a and b, or a cyclically alternating word $t_1 t_2 \cdots t_s$ in non-zero powers of c and non-empty square-free words in a and b with the property that for each factor t_i which is a positive power of c, t_{i-1} ends with $b^{\pm 1}$ and t_{i+1} starts with $a^{\pm 1}$, while for each factor t_i which is a negative power of c, t_{i-1} ends with $a^{\pm 1}$ and t_{i+1} starts with $b^{\pm 1}$.

EXAMPLE. $s = \{c^6, a^{-1}baba^{-1}b^{-1}, (a^{-1}baba^{-1})c^{-5}(ba^{-1}ba)c^{-4}(b^{-1}a^{-1}b^{-1})c^3\}$.

3.3.3. Each element of s is a cyclically alternating $(\{a, a^{-1}, b, b^{-1}\}, \{c, c^{-1}\})$-word.

EXAMPLE. $s = \{cac^{-1}b^{-1}c^{-1}acb\}$.

3.3.4. Each element of s is either a power of c, or a cyclically square-free word in a and b, or a cyclically alternating word $t_1 t_2, \cdots t_s$ in non-zero powers of c and non-empty square-free words in a and b with the property that for each factor t_i which is a positive power of c, t_{i-1} ends with a and t_{i+1} starts with a, while for each factor t_i which is a negative power of c, t_{i-1} ends with a^{-1} and t_{i+1} starts with a^{-1}.

EXAMPLE. $s = \{c^6, a^{-1}baba^{-1}b^{-1}, (abab^{-1}a^{-1})c^{-5}(a^{-1}b^{-1}aba^{-1})c^{-4}(a^{-1}ba)c^3\}$.

3.3.5. Each element of s is either a cyclically alternating $(\{a\}, \{b, c, bc\})$-word, or a cyclically alternating $(\boldsymbol{u}, \boldsymbol{v})$-word, where $\boldsymbol{u}$ is the set of alternating $(\{a\}, \{b, c, bc\})$-words and $\boldsymbol{v}$ is the set of inverses of words in $\boldsymbol{u}$. In addition, it is required in the latter case that no $\boldsymbol{u}$-factor begins with c or ends with b, and no $\boldsymbol{v}$-factor begins with b^{-1} or ends with c^{-1}.

EXAMPLE. $s = \{ababcac, (ababc)(bababca)^{-1}(abaca)(babc)^{-1}\}$.

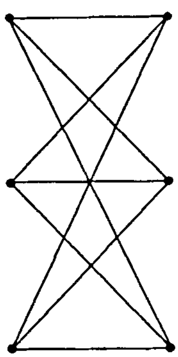
3.3.6. Each element of s is either a power of a, or a power of c, or a cyclically alternating $(\boldsymbol{u}, \boldsymbol{v})$-word, where $\boldsymbol{u} = \{b, b^{-1}\}$ and $\boldsymbol{v}$ is the set of words of the form $(a^m c^{-n})^{\pm 1}$ (m, n positive integers).

EXAMPLE. $s = \{a^{10}, c^6, (a^2c^{-3})b(ac^{-5})^{-1}b(a^7c^{-2})^{-1}b^{-1}\}$.

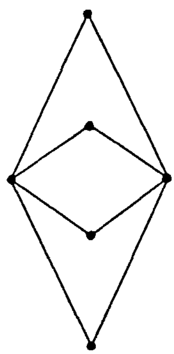
3.3.7. Each element of s is either a power of ac, or a cyclically alternating (u, v)-word, where $u = \{b, b^{-1}\}$ and v is the set of words of the form PQ^{-1} (P, Q non-empty positive square-free words in a and c).

EXAMPLE. $s = \{(ac)^4, [(ac)^2(a^{-1}c^{-1})^3a^{-1}]b[aca(c^{-1}a^{-1})]b^{-1}\}$.

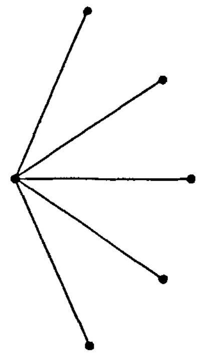
To prove the above, observe that $\mathcal{P}$ satisfies $T(4)$ if and only if $\Gamma(\mathcal{P}^{st})$ is a subgraph of a labelling by $\{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$ of one of the following:



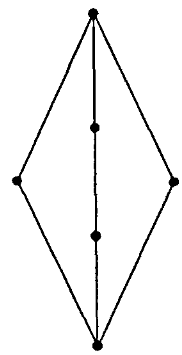
$K_{3,3}$



$K_{2,4}$

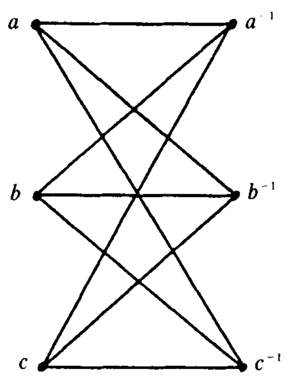


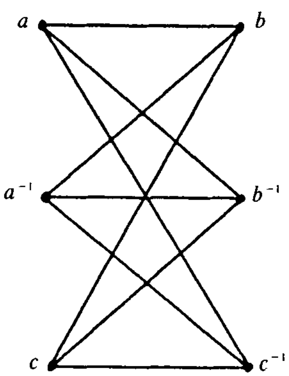
$K_{1,5}$



Γ_0

Up to Ω_3-equivalence, the labellings of $K_{3,3}$ are:

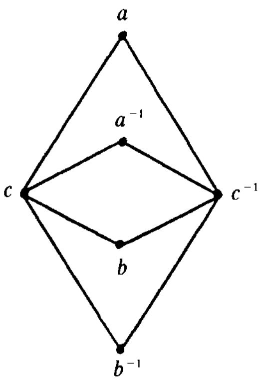


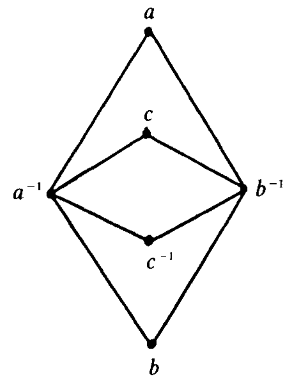


Now $\Gamma(\mathcal{P}^{st})$ will be a subgraph of the former labelling if and only if each relator of $\mathcal{P}$ is positive or negative; $\Gamma(\mathcal{P}^{st})$ will be a subgraph of the latter labelling if and only if r is the symmetrized closure of a set s satisfying 3.3.2.

We will say that $\mathcal{P}$ is of *Type A* if $\Gamma(\mathcal{P}^{st})$ is Ω_3-equivalent to a subgraph of one of the above labellings of $K_{3,3}$.

Up to Ω_3-equivalence, the labellings of $K_{2,4}$ are:





Now $\Gamma(\mathscr{P}^{st})$ will be a subgraph of the former labelling if and only if r is the symmetrized closure of a set s satisfying 3.3.3. We will show that if $\Gamma(\mathscr{P}^{st})$ is a subgraph of the latter labelling then $\mathscr{P}$ is already of Type A. By 2.3.2 we can assume that $\mathscr{P}$ is finite, so that we can make use of 2.3.3. Looking at the weights of the vertices a, a^{-1}, b, b^{-1} we obtain:

$$m_{a,a^{-1}} + m_{a,b^{-1}} = m_{a^{-1},a} + m_{a^{-1},c} + m_{a^{-1},c^{-1}} + m_{a^{-1},b},$$

$$m_{b,a^{-1}} + m_{b,b^{-1}} = m_{b^{-1},a} + m_{b^{-1},c} + m_{b^{-1},c^{-1}} + m_{b^{-1},b}.$$

These give

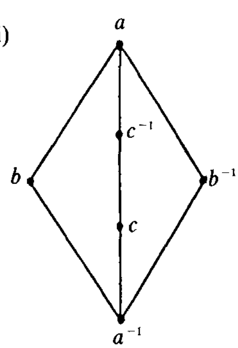
$$0 = m_{a^{-1},c} + m_{a^{-1},c^{-1}} + m_{b^{-1},c} + m_{b^{-1},c^{-1}},$$

and so $m_{a^{-1},c} = m_{a^{-1},c^{-1}} = m_{b^{-1},c} = m_{b^{-1},c^{-1}} = 0$. Thus the edges $\{a^{-1}, c\}$, $\{a^{-1}, c^{-1}\}$, $\{b^{-1}, c\}$, $\{b^{-1}, c^{-1}\}$ do not belong to $\Gamma(\mathscr{P}^{st})$.

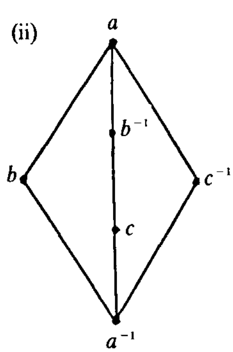
An argument like that used for two-generator presentations above shows that if $\Gamma(\mathscr{P}^{st})$ is a subgraph of a labelling of $K_{1,5}$ then $\mathscr{P}$ is of Type A. (Alternatively, one can use a weight argument.)

Up to Ω_3-equivalence, the labellings of Γ_0 are:

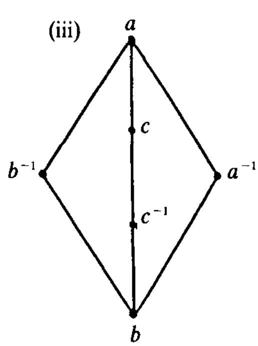
(i)

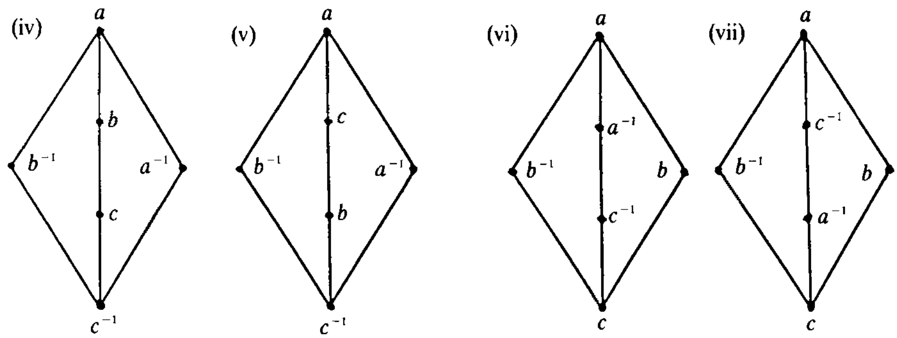


(ii)



(iii)





Now $\Gamma(\mathcal{P}^{st})$ will be a subgraph of (i) (resp. (ii), (vi), (vii)) if and only if r is the symmetrized closure of a set s satisfying 3.3.4 (resp. 3.3.5, 3.3.6, 3.3.7). We now show that if $\Gamma(\mathcal{P}^{st})$ is a subgraph of one of (iii), (iv), (v), then $\mathcal{P}$ is already of Type A. As before, we can assume that $\mathcal{P}$ is finite.

Consider (iii). Looking at the weights of a, a^{-1} and b, b^{-1} and using 2.3.3 we have:

$$m_{a,b^{-1}} + m_{a,c} + m_{a,a^{-1}} = m_{a^{-1},a} + m_{a^{-1},b},$$

$$m_{b,a^{-1}} + m_{b,c^{-1}} + m_{b,b^{-1}} = m_{b^{-1},b} + m_{b^{-1},a}.$$

From these we obtain

$$m_{a,c} + m_{b,c^{-1}} = 0.$$

Thus $m_{a,c} = m_{b,c^{-1}} = 0$, so the edges $\{a,c\}$, $\{b,c^{-1}\}$ do not belong to $\Gamma(\mathcal{P}^{st})$.

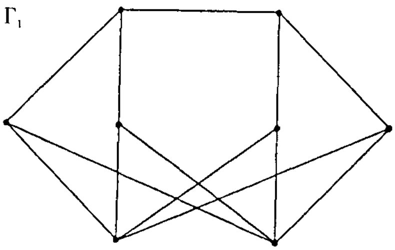
Now consider (iv) and (v). Looking at the weights of a and a^{-1}, b and b^{-1}, c and c^{-1}, and proceeding similarly as above we obtain that $m_{a,b} = 0$ (for (iv)), $m_{b,c^{-1}} = 0$ (for (v)).

3.4. n-Generator Presentations

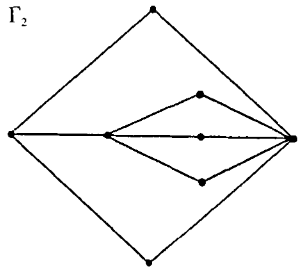
One can obviously, in theory, use arguments similar to those above to determine, for any n, all n-generator presentations satisfying $T(4)$. Firstly one must compute all graphs on $2n$ vertices without triangles, and which are maximal with this property. (We remark that, by Turán's Theorem [1, p. 17], a graph on $2n$ vertices without triangles has at most n^2 edges.) Then one must analyse the labellings of these graphs by $\{x_1,\dots,x_n,\ x_1^{-1},\dots,x_n^{-1}\}$, making use of weight arguments as above.

Unfortunately, even for $n = 4$, the amount of labour involved in carrying out these computations by hand is prohibitive. There are 10 graphs on 8 vertices without triangles, and which are maximal with this property, namely: the bipartite graphs $K_{4,4}$, $K_{3,5}$, $K_{2,6}$, $K_{1,7}$, and the graphs $\Gamma_1, \ldots, \Gamma_6$ depicted below. Some of these graphs are easily analysed, but others (such as Γ_2) have a large number of labellings (up to Ω_4-equivalence).

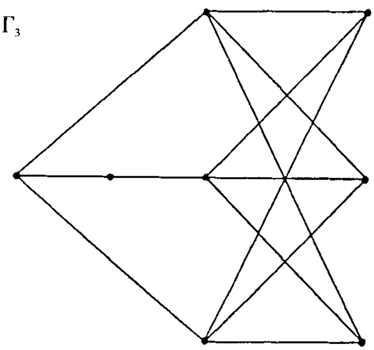
Γ_1



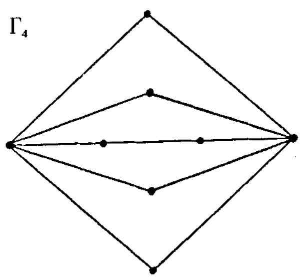
Γ_2



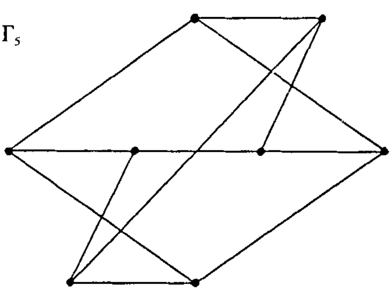
Γ_3



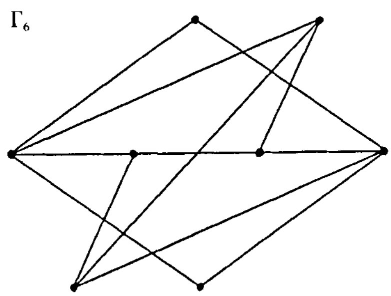
Γ_4



Γ_5



Γ_6



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